Mathematics of Random Systems Summer School

11-15 September 2023

Research Institute for Mathematical Sciences (RIMS), Kyoto University

Organisers: Thomas Cass (Imperial College London), David Croydon (Kyoto University), Ben Hambly (University of Oxford), Hiroshi Kokubu (Kyoto University), Seiichiro Kusuoka (Kyoto University), Jeroen Lamb (Imperial College London)

Monday 11th September	Tuesday 12th September	Wednesday 13th September	Thursday 14th September	Friday 15th September
09:30–10:30 Etheridge	09:30–10:30 Etheridge	09:30–10:30 Blumenthal	09:30–10:30 Blumenthal	09:20–10:20 Etheridge
10:50–12:10 Fukushima	10:50–12:10 Fukushima	10:50–12:10 Fukushima	10:50–11:50 Etheridge	10:40–12:00 Student talks
14:00–14:40 Zeitouni	14:00–14:40 Sumi		14:00–14:40 Sakai	12:10–12:50 Yano
15:00–16:20 Student talks	15:00–16:20 Student talks		15:00–16:30 Student talks	12:50–13:00 Closing
16:30–17:30 Blumenthal	16:30–17:30 Blumenthal		16:40–17:20 Kawabi	

Lecture courses

Alex Blumenthal . . . . . . *Chaos, randomness and fluid dynamics*

Alison Etheridge . . . . . . *Some mathematical models from population genetics*

Ryoki Fukushima . . . . . . *Concentration inequalities in random media*

Invited talks

Ofer Zeitouni . . . . . . *Some voting schemes and nonlinear diffusion-reaction equations*

Hiroki Sumi . . . . . . *Random dynamical systems of polynomial automorphisms on* $\mathbb{C}^2$

Akira Sakai . . . . . . *Mathematical foundation of various MCMC methods*

Hiroshi Kawabi . *Central limit theorems for non-symmetric random walks on nilpotent covering graphs*

Kouji Yano . . . . . . *Arcsine and Darling–Kac laws for piecewise linear random interval maps*

Student/Postdoc talks

Monday	*Tuesday*	*Thursday*	*Friday*
Tomohiro Aya	Masahisa Ebina	Sarah-Jean Meyer	Akshunna S. Dogra
Bernat Bassols Cornudella	Kohei Hayashi	Hirotatsu Nagoji	Anh Duc Vu
Matthew Buckland	Timothy Kang	Ryoichiro Noda	Satomi Watanabe
William Turner	Jonas Köppl	Takumu Ooi	Xianliang Zhao
		Daiki Tagami	

www.randomsystems-cdt.ac.uk/event/mathematics-of-random-systems-summer-school-2023

Lecture courses

Alex Blumenthal (Georgia Institute of Technology)

Chaos, randomness and fluid dynamics

By now there is a well-developed abstract theory for the description of chaotic phenomena in deterministic dynamical systems, including both ergodic-theoretic tools such as the multiplicative ergodic theorem and Pesin stable/unstable manifold theory. However, applying this theory to systems of practical interest remains a notoriously challenging problem, as many instances of chaotic behavior are structurally unstable and intermingled in a complicated way – in both phase space and parameter space – with "ordered" asymptotically periodic dynamics.

The topic of this minicourse is the remarkable way in which external noisy perturbation "regularizes" this complicated dynamical picture: when subjected to mildly nondegenerate noise, chaotic regimes are far more tractable to study: regimes are structurally stable (with respect to small changes of fixed parameters of the model); numerical estimates are far more reliable; and one can rigorously prove the presence of chaotic behavior far beyond what can be done in the absence of noise.

The first two talks will cover the basics: the definition of a random dynamical system, elements of ergodic theory, an introduction to deterministic chaos, and some known examples of pathological behavior. In the second two talks, I will hone in on more specific methods for verifying chaotic behavior of volume-preserving and weakly-dissipated stochastic diffusions, both of which have applications to fluid mechanical systems (joint work with J. Bedrossian, S. Punshon-Smith, M. Coti-Zelati, and R. Gvalani).

Alison Etheridge (University of Oxford)

Some mathematical models from population genetics

Our aim in these lectures is to provide some flavour of the rich variety of mathematical models that arise in the study of the genetics of natural populations and to present some of the main tools that are used to explore their behaviour. Although our main focus is spatially distributed populations, we begin with an introduction to the classical Wright-Fisher and Moran models. This provides the opportunity to introduce some terminology and gain some insight into the nature of the approaches that we'll take in the more complicated models that lie ahead. Key concepts will be diffusion approximations, genealogical trees, and the notion of duality.

The next step will be to generalise these models to include spatial structure. This topic has a long and distinguished history, and again we shall begin with classical models due to Kimura, Wright, and Malécot, in which very explicit calculations are possible. However, not only do these models make some very strong assumptions, but in the case of the Wright and Malécot approach to modelling populations evolving in a spatial continuum, in what is arguably the most natural setting of a population evolving in continuous time in a continuous two-dimensional spatial environment, there is a mathematical inconsistency, stemming from what is popularly known as 'the pain in the torus'. As time permits, we shall introduce a variety of models that overcome some of these issues, and use them to investigate the way in which spatial structure interacts with simple forms of natural selection.

Ryoki Fukushima (University of Tsukuba)

Concentration inequalities in random media

Concentration inequalities provide bounds on the probability that a random variable deviates from its mean value or median. They originate from the study of the sum of independent random variables but modern formulations are applicable to many other problems. In the first lecture, I illustrate their use in random media through an application to the directed polymer in a bounded random environment. In the second and third lectures, I explain two examples from random media in which the assumptions of concentration inequality do not hold but one can still use them with the help of model-dependent tricks. The first one is the Anderson Hamiltonian in a large finite box. The second one is the number of open paths in oriented percolation.

Invited talks

Ofer Zeitouni (Weizmann Institute of Science)

Some voting schemes and nonlinear diffusion-reaction equations

Etheridge, Freeman and Penington have proposed a probabilistic interpretation of certain classes of reaction-diffusion equations which are not of the F-KPP type, such as the Allen-Cahn equation. An, Henderson and Ryzhik showed how this idea applies to a wide class of such nonlinear equations. I will describe some of these connections in the discrete setup, especially when log-concavity of the distribution of increments is not presented. I will focus of questions of tightness. Joint work with Xaver Kriechbaum and Lenya Ryzhik.

Hiroki Sumi (Kyoto University)

Random dynamical systems of polynomial automorphisms on $\mathbb{C}^2$

We consider i.i.d. random dynamical systems of polynomial automorphisms on $\mathbb{C}^2$. In particular, we consider i.i.d. random dynamical systems of complex Henon maps and their conjugated maps on $\mathbb{C}^2$. We show that for a generic such system, we have the following.

(1) There exist only finitely many minimal sets $L_1, \ldots, L_m$ in $\mathbb{C}^2$, and each L_j is attracting.

(2) For each initial value z in $\mathbb{C}^2$ and for almost every sequence of maps $\gamma = (\gamma_1, \gamma_2, \ldots)$, the orbit $\{\gamma_n \cdots \gamma_1(z)\}_{n=1}^{\infty}$ tends to a point in the line at infinity or tends to one of $L_1, \ldots, L_m$.

(3) For each initial value z in $\mathbb{C}^2$ and for almost every sequence of maps $\gamma = (\gamma_1, \gamma_2, \ldots)$, the Lyapunov exponent of γ at z with respect to the Fubini-Study metric on $\mathbb{P}^2$ is negative.

Note that the above phenomenon cannot hold for deterministic dynamical systems of iterations of single complex Henon maps. Thus we see a randomness-induced phenomenon (a phenomenon of random dynamical systems which cannot hold for deterministic dynamical systems of iterations of single complex Henon maps).

Akira Sakai (Hokkaido University)

Mathematical foundation of various MCMC methods

Combinatorial optimization problems are ubiquitous in various fields of practical and theoretical interest. The famous traveling salesman problem is one of them. One approach to tackle those problems is to translate them into Ising models whose Hamiltonian H takes its minimum at a spin configuration (= a ground state) which corresponds to an optimal solution to the corresponding original problem. Standard MCMC methods, such as the Glauber dynamics and the Metropolis algorithm, have been used for decades to sample the Gibbs distribution, which is proportional to $e^{-H/T}$, hence close to the uniform distribution over the ground states when the temperature T is very small. However, those MCMC methods are based on single-spin flip rules, hence prone to being slow.

In this talk, I will explain three other MCMC methods, two among which are based on multi-spin flip rules, hence potentially fast. I will show several mathematical results, as well as numerical results to compare which is better in which context.

This talk is based on joint work with Bruno Hideki Fukushima-Kimura and many others involved in the CREST project for the past five years.

Hiroshi Kawabi (Keio University)

Central limit theorems for non-symmetric random walks on nilpotent covering graphs

The long time asymptotics for random walks on infinite graphs is a principal topic in both geometry and probability theory. A covering graph of a finite graph with a nilpotent covering transformation group is called a nilpotent covering graph, regarded as a generalization of a crystal lattice or the Cayley graph of a finite generated group of polynomial growth.

In this talk, we discuss non-symmetric random walks on nilpotent covering graphs from a view point of the theory of discrete geometric analysis developed by Kotani and Sunada, and give central limit theorems for them. We also mention a relationship between the limiting diffusion and (the Lyons lift of) distorted Brownian rough path.

This talk is based on joint work with Satoshi Ishiwata (Yamagata University) and Ryuya Namba (Kyoto Sangyo University).

Kouji Yano (Osaka University)

Arcsine and Darling–Kac laws for piecewise linear random interval maps

The Boole transformation induces a deterministic dynamical system with two indifferent fixed points where the occupation times around them satisfy the arcsine and Darling–Kac laws which are well-known for the one-dimensional simple symmetric random walk. We give examples of piecewise linear random interval maps satisfying the arcsine and Darling–Kac laws. They are constructed by random switch of two piecewise linear maps with attracting or repelling fixed points, which behave as if they were indifferent fixed points of a deterministic map.

Student/Postdoc talks

Tomohiro Aya (Kyoto University)

Convergence rates in stochastic homogenization of elliptic and parabolic equations with unbounded coefficients

Quantitative stochastic homogenization is a field that aims to obtain a rate of the convergence in stochastic homogenization under the assumption of quantitative mixing conditions in ergodicity. In this talk, we consider stochastic homogenization of elliptic and parabolic equations with unbounded and non-uniformly elliptic coefficients. Extending subadditive arguments, we get an estimate for the rate of the convergence of the solution of the Cauchy-Dirichlet problem under the condition that coefficients in the unit cube have a certain exponential integrability.

Bernat Bassols Cornudella (Imperial College London)

Noise-induced chaos: a conditioned random dynamics perspective

We consider transitions to chaos in random dynamical systems induced by an increase of noise amplitude. We show how the emergence of chaos (indicated by a positive Lyapunov exponent) in a logistic map with bounded additive noise can be analysed in the framework of conditioned random dynamics through expected escape times and conditioned Lyapunov exponents for a compartmental model representing the competition between contracting and expanding behaviour. We find that the noise-induced transition to chaos is caused by a rapid decay of the expected escape time from the contracting compartment, while all other order parameters remain approximately constant.

Matthew Buckland (University of Oxford)

Branching interval partition diffusions

We construct an interval-partition-valued diffusion from a collection of excursions sampled from the excursion measure of a real-valued diffusion, and we use a spectrally positive Lévy process to order both these excursions and their start times. At any point in time, the interval partition generated is the concatenation of intervals where each excursion alive at that point contributes an interval of size given by its value. Previous work by Forman, Pal, Rizzolo and Winkel considers self-similar interval partition diffusions – and the key aim of this work is to generalise these results by dropping the self-similarity condition. The interval partition can be interpreted as an ordered collection of individuals (intervals) alive that have varying characteristics and generate new intervals during their finite lifetimes, and hence can be viewed as a class of Crump-Mode-Jagers-type processes.

Akshunna S. Dogra (Imperial College London)

ν-tangent kernels

The approximation and generalization capacity of machine learning models has been profitably leveraged across a staggeringly wide variety of tasks. In particular, appropriately initialized Neural Networks sampled from suitable functional spaces invariably find stages of exponential learning. We introduce ν - Tangent Kernels (νTKs), functional analytic objects partly inspired from the Neural Tangent Kernel (NTK), to build a generic theory for Neural Network optimization and generalization. Specifically, we prove that for a large category of sufficiently well-posed and controlled problems, appropriately initialized Neural Network based models are capable of exponentially learning/solving the tasks/problems at hand. Notably, these results are showcased for a much wider class of loss functions/architectures than the standard mean squared error/large width regime that is usually the focus of conventional NTK analysis. The analysis applies to diverse practical problems solved using real networks such as differential equation solvers, shape recognition, classification, feature extraction, etc. We exemplify the power of the νTK perspective by demonstrating the strong agreement between the predictions made from within this theory and the empirically observed optimization profiles, across different regimes and problems.

Masahisa Ebina (Kyoto University)

Central limit theorems for high-dimensional stochastic wave equations

This talk considers stochastic wave equations in spatial dimensions $d \geq 4$. The driving noise is assumed to be given by a Gaussian noise that is white in time and has the Riesz kernel as a spatial correlation. I will explain how to obtain the central limit theorems for the spatial average of the solution.

Kohei Hayashi (RIKEN iTHEMS)

Anomalous behavior in the Bernardin-Stoltz model

We study a microscopic model of interacting oscillators, which admits two conserved quantities, volume, and energy. We begin with a system driven by a general nonlinear potential under high-temperature regime by taking the inverse temperature of the system asymptotically small. As a consequence, one can extract a principal part (by a simple Taylor expansion argument), which is driven by the harmonic potential, and we show that previous results for the harmonic chain are covered with generality. We consider two fluctuation fields, which are defined as a linear combination of the fluctuation fields of the two conserved quantities, volume, and energy, and we show that the fluctuations of one field converge to a solution of the stochastic Burgers equation, in a stronger asymmetric regime, whereas a fractional diffusion equation given by a skewed Lévy process from the fluctuations of the other field.

Timothy Kang (Imperial College London)

An algorithm for the Rao decomposition of a quasimartingale

In the 1930s, the mathematician Guido Ascoli introduced an algorithm which returns the decomposition of a finite variational function $f : [a, b] \to \mathbb{R}$ as the difference of two non-decreasing functions. In this talk, we discuss how to modify Ascoli's algorithm to obtain an algorithm which returns the Rao decomposition of a quasimartingale. After introducing basic facts about quasimartingales and Ascoli's algorithm, along with its properties, we present our modification of Ascoli's algorithm. We explain the intuition through which we have conjectured our modification of Ascoli's algorithm and briefly outline a proof that our conjecture returns the Rao decomposition of a quasimartingale. We present a few examples of how to apply our algorithm and discuss future work.

Jonas Köppl (Weierstrass Institute Berlin and TU Berlin)

On the long-time behaviour of reversible interacting particle systems in one and two spatial dimensions

Interacting particle systems are countable systems of locally interacting Markov processes and are often used as toy models for stochastic phenomena with an underlying spatial structure. Even though the definition of an interacting particle system often looks very simple, it is in general surprisingly difficult to say anything non-trivial about their behavior. Via a Lyapunov-functional approach, we show that, under quite general assumptions on the dynamics, the attractor of a (possibly non-translation-invariant) interacting particle system in one or two spatial dimensions is contained in the set of Gibbs measures if the dynamics admits a reversible Gibbs measure. In particular, this implies that there can be no reversible interacting particle system that exhibits time-periodic behaviour and that every reversible interacting particle system is ergodic if and only if the reversible Gibbs measure is unique. In the special case of non-attractive stochastic Ising models this answers a question due to Liggett.

Sarah-Jean Meyer (University of Oxford)

A FBSDE approach to Euclidean quantum field theories

We propose a different approach to stochastic quantisation via a novel kind of forward-backward stochastic differential equation. Compared to the more standard stochastic quantisation equations, this representation yields a pathwise scale-by-scale coupling with the Gaussian free field which, if it can be controlled in the UV limit, enables tight control over the resulting theory. As a result, it allows for a more detailed study beyond existence. In example of the sine-Gordon model, if time allows, we sketch how this allows to show exponential decay of correlations using a simple coupling argument.

Hirotatsu Nagoji (Kyoto University)

Normalizability of the Gibbs measures associated with multivariate version of $P(\Phi)_2$ model

We consider the Gibbs measures associated with multivariate version of $P(\Phi)_2$ quantum field model on the torus. We observe the (non-)normalizability of the measures by the variational method introduced by Barashkov and Gubinelli. We also consider some variants of the model.

Ryoichiro Noda (Kyoto University)

Convergence of local times of stochastic processes associated with resistance forms

A resistance metric on a space characterizes, as the electrical energy, a corresponding bilinear form (a resistance form) and, combined with a measure on the space (and under certain technical conditions), determines uniquely a Dirichlet form and a stochastic process on the space. Croydon-Hambly-Kumagai (2017) showed that if a sequence of spaces equipped with resistance metrics and measures converge with respect to the Gromov-Hausdorff-vague topology and a uniform volume doubling (UVD) condition is satisfied, then the associated stochastic processes and local times also converge. However, the UVD condition is too strong for many sequences of random graphs. In the subsequent work of Croydon (2018), the UVD condition was relaxed and the convergence of the processes was established under a weaker non-explosion condition. However, the convergence of local times was left open. In this talk, we show that if the spaces additionally satisfy a certain metric-entropy condition, which is weaker than the UVD condition, then the local times of the processes also converge. The metric-entropy condition can be checked using volume estimates of balls in the spaces, and we also present some example applications.

Takumu Ooi (Kyoto University)

Convergence of processes time-changed by GMC

As represented by the Liouville measure, Gaussian multiplicative chaos (GMC) is a random measure constructed from a Gaussian field. Under certain technical assumptions, we prove the convergence of a process time-changed by GMC in the case the latter object is square integrable (the L^2-regime). As examples of the main result, we prove that, in the whole L^2-regime, the scaling limit of the Liouville simple random walk on $\mathbb{Z}^2$ is Liouville Brownian motion and, as $\alpha \to 1$, Liouville α-stable processes on $\mathbb{R}$ converge weakly to the Liouville Cauchy process.

Daiki Tagami (University of Oxford)

tstrait: a genome-wide simulator of quantitative traits

tstrait is a new open-source Python library that incorporates a wide range of state-of-the-art algorithms for simulating quantitative traits from the ancestral recombination graph (ARG). ARG is an active field of research in population genetics and statistical genetics due to its promising aspects in genome-wide association study (GWAS). This package can simulate quantitative traits at a faster computational speed than conventional methods, and this serves as an essential infrastructure for utilizing ARG in GWAS simulations. We will provide an overview of the simulation framework that is incorporated in tstrait and highlight some recent developments in ARG.

William Turner (Imperial College London)

Generative modelling of Lévy area for high order SDE simulation

It is well known that, when numerically simulating solutions to SDEs, achieving a strong convergence rate better than $O(\sqrt{h})$ (where h is the step-size) requires the use of certain iterated integrals of Brownian motion, commonly referred to as its “Lévy areas”. However, these stochastic integrals are difficult to simulate due to their non-Gaussian nature and for a d-dimensional Brownian motion with $d > 2$, no fast almost-exact sampling algorithm is known.

We propose LévyGAN, a deep-learning-based model for generating approximate samples of Lévy area conditional on a Brownian increment. Our generator employs a tailored GNN-inspired architecture, which enforces the correct dependency structure between the output distribution and the conditioning variable. We also incorporate a novel training mechanism termed “Chen-training”, which circumvents the need for expensive-to-generate training datasets. This new training procedure is underpinned by two theoretical results.

For 4-dimensional Brownian motion, we show that LévyGAN exhibits state-of-the-art performance across several metrics which measure both the joint and marginal distributions. We conclude with a numerical multilevel Monte-Carlo experiment on the log-Heston model, a popular SDE in mathematical finance.

Anh Duc Vu (Weierstrass Institute Berlin)

A long-range contact process in random environment

We study a discrete time contact process on an infinitely long street ($\mathbb{Z}$). During the day, an infection may spread to other houses (vertices) with probability $d^{-\alpha}$ where d is the distance between the houses and $\alpha > 1$. At night, the vertices may recover again with probability $1 - p$.

The infection is obstructed in 2 ways: First, not all vertices are susceptible since houses might not be located on every vertex. A sparse environment then helps in suppressing further spreading. Furthermore, global lockdowns may occur during which the infection cannot spread.

We show that the infection may still survive as long as p is large enough.

Satomi Watanabe (Kyoto University)

On simple random walk on a high-dimensional loop-erased random walk

Heat kernel estimate is a key to understand properties of random walks on graphs. In this talk, I will talk about a result on annealed heat kernel estimates for simple random walk on a loop-erased random walk on $\mathbb{Z}^d$ where $d \geq 5$. I will also introduce an averaged local central limit theorem, which leads to the heat kernel result.

Xianliang Zhao (Bielefeld University)

Mean-field limit of non-exchangeable interacting diffusions with singular kernels

In the talk, we shall discuss the mean-field limits of interacting diffusions without exchangeability. The results apply to the 2D stochastic vortex model with general intensities, related to the 2D Navier-Stokes equation, and graphon particle systems with singular kernels. Our method is based on a tightness argument and uses the systems' uniform Fisher information. The talk is based on the joint work with Zhenfu Wang and Rongchan Zhu and my thesis.

Conference Location

Access information for the Research Institute for Mathematical Sciences (RIMS) can be found here:

`www.kurims.kyoto-u.ac.jp/en/access-01.html`.

See also:

`www.kyoto-u.ac.jp/en/access/north-campus-map`.

Source: `www.kyoto-u.ac.jp/en/access/campus-maps-for-download`

All talks will take place in room 420 of RIMS (no. 7 on the above campus map).

Lunch can be found in the cafeteria shown on the above campus map. There are also various restaurants located close to RIMS on Imadegawa Street.